Midterm Intro to Logic - 13 Dec. 2017 - Model Answers

1 a. Translation key:

O : the human body has enough oxygen
B : the heart pumps blood around
L : the lungs work

Translation: $\neg O \vee (B \wedge L)$

also correct is the logically equivalent: $\neg(B \wedge L) \rightarrow \neg O$

b. A : you are ambitious
C : you want to go for a 'cum laude' bachelor's degree
E : you have to get at least an 8 on average
R : you can take re-sit exams

Translation: $(A \wedge C) \rightarrow (E \wedge \neg R)$

2 Translation key:

a: Aron   b: Bas   c: Chris   d: Dunya

$H(x,y)$: x hates y
$L(x,y)$: x loves y
$S(x,y)$: x and y are siblings

a) $(L(a,c) \wedge L(b,c)) \leftrightarrow (\neg H(c,a) \wedge \neg H(c,b))$. Also correct:
$(L(a,c) \wedge L(b,c)) \leftrightarrow \neg(H(c,a) \vee H(c,b))$

b) $(L(c,a) \vee (L(d,a) \wedge L(b,a))) \rightarrow S(a,b)$

It would also be correct to use a predicate like $G(x,y)$: x is sibling to y. Then the translation would be:
$(L(c,a) \vee (L(d,a) \wedge L(b,a))) \rightarrow (G(a,b) \wedge G(b,a))$

3 a)

1.	$A \to C$	
2.	$\neg D \to \neg B$	
3	$A \lor B$	
4	A	
5	C	$\to$ Elim: 1, 4
6	$C \lor D$	$\lor$ Intro: 5
~~7~~ 7	B	
8	$\neg D$	
9	$\neg B$	$\to$ Elim: 2, 8
10	$\bot$	$\bot$ Intro: 7, 9
11.	$\neg\neg D$	$\neg$ Intro: 8-10
12	D	$\neg$ Elim: 11
13	$C \lor D$	$\lor$ Intro: 12
14	$C \lor D$	$\lor$ Elim: 3, 4-6, 7-13
15	$(A \lor B) \to (C \lor D)$	$\to$ Intro: 3-14

3 b)

1.	$\neg(a = a \land P(b))$	
2.	$a = b$	
3	$P(a)$	
4	$P(b)$	$=$ Elim: 3, 2
5	$a = a$	$=$ Intro
6.	$a = a \land P(b)$	$\land$ Intro: 5, 4
7	$\bot$	$\bot$ Intro: 6, 1
8	$\neg P(a)$	$\neg$ Intro: 3-7

3 c)

1	P	
2	$\neg P$	
3	P	Reit: 1
4.	$\neg P \rightarrow P$	$\rightarrow$ Intro: 2-3
5	$\neg P \rightarrow P$	
6	$\neg P$	
7	P	$\rightarrow$ Elim: 5, 6
8	$\perp$	$\perp$ Intro: 7, 6
9	$\neg\neg P$	$\neg$ Intro: 6-8
10	P	$\neg$ Elim: 9
11	$P \leftrightarrow (\neg P \rightarrow P)$	$\leftrightarrow$ Intro: 1-4, 5-10

4 a) $((Q \leftrightarrow \neg R) \vee (P \leftrightarrow \neg Q)) \vee (\neg P \leftrightarrow \neg R)$

P	Q	R	$Q \leftrightarrow$	$\neg R$	$\vee$	$P \leftrightarrow$	$\neg Q$	$\vee$	$\neg P$	$\leftrightarrow$	$\neg R$
T	T	T	F	F	F	F	F	T	F	T	F
T	T	F	T	T	T	F	F	T	F	F	T
T	F	T	T	F	T	T	T	T	F	T	F
T	F	F	F	T	T	T	T	T	F	F	T
F	T	T	F	F	T	T	F	T	T	F	F
F	T	F	T	T	T	T	F	T	T	T	T
F	F	T	T	F	T	F	T	T	T	F	F
F	F	F	F	T	F	F	T	T	T	T	T

Conclusion: because the formula is true in all 8 rows of the truth table, it is indeed a tautology (tautological truth)

4b.

	$a=b$	Medium(a)	Large(b)	Medium(a) $\leftrightarrow \neg$Large(b)
1)	T	T	T	F F
2)	T	T	F	T T
3)	T	F	T	T F
4)	T	F	F	F T
5)	F	T	T	F F
6)	F	T	F	T T
7)	F	F	T	T F
8)	F	F	F	F T

- Row 1 is spurious
- Medium(a) $\leftrightarrow \neg$Large(b) is <u>not</u> a logical consequence of $a=b$, because there is a non-spurious row, namely row 4, in which $a=b$ is true, but Medium(a) $\leftrightarrow \neg$Large(b) is false.

5a)

$$\neg(P \leftrightarrow Q) \to \neg\neg(P \to R) \quad \overset{\neg\neg}{\Longleftrightarrow}$$
$$\neg(P \leftrightarrow Q) \to (P \to R) \quad \overset{\text{def} \leftrightarrow}{\Longleftrightarrow}$$
$$\neg((\neg P \lor Q) \land (\neg Q \lor P)) \to (P \to R) \quad \overset{\text{def.} \to,\ 2\times}{\Longleftrightarrow}$$
$$\neg\neg((\neg P \lor Q) \land (\neg Q \lor P)) \cancel{\to} \lor (\neg P \lor R) \quad \overset{\neg\neg}{\Longleftrightarrow}$$
$$((\neg P \lor Q) \land (\neg Q \lor P)) \lor (\neg P \lor R)$$

This sentence is in NNF

5b)

$$\neg(P \lor R) \lor (Q \land (P \to S)) \quad \overset{\text{de Morgan}}{\Longleftrightarrow}$$
$$(\neg P \land \neg R) \lor (Q \land (P \to S)) \quad \overset{\text{def} \to}{\Longleftrightarrow}$$
$$(\neg P \land \neg R) \lor (Q \land (\neg P \lor S)) \quad \overset{\text{distrib.}}{\Longleftrightarrow}$$
$$((\neg P \land \neg R) \lor Q) \land [(\neg P \land \neg R) \lor (\neg P \lor S)] \quad \overset{\text{distr, } 2\times}{\Longleftrightarrow}$$
$$(\neg P \lor Q) \land (\neg R \lor Q) \land (\neg P \lor S \lor \neg P) \land (\neg P \lor S \lor \neg R)$$

The latter formula is in CNF (though can be simplified)

6).

a	T	f	T
b	T	g	F
c	T	h	T
d	F	i	F
e	T	j	F

Bonus

7

1 $\neg\big((P \to Q) \lor (Q \to R) \lor (R \to P)\big)$

2 P

3 R

4 P Reit : 2

5 $R \to P$ $\to$ Intro : 3-4

6 $(P \to Q) \lor (Q \to R) \lor (R \to P)$ $\lor$ Intro : 5

7 $\bot$ $\bot$ Intro : 6, 1

8 $\neg P$ $\neg$ Intro : 2-7

9 P

10 $\bot$ $\bot$ Intro : 9, 8

11 Q $\bot$ Elim : 10

12 $P \to Q$ $\to$ Intro : 9-11

13 $(P \to Q) \lor (Q \to R) \lor (R \to P)$ $\lor$ Intro : 12

14 $\bot$ $\bot$ Intro : 13, 1

15 $\neg\neg\big((P \to Q) \lor (Q \to R) \lor (R \to P)\big)$ $\neg$ Intro : 1-14

16 $(P \to Q) \lor (Q \to R) \lor (R \to P)$ $\neg$ Elim : 15

Many alternative correct proofs have been submitted, e.g. first proving $Q \lor \neg Q$, and then showing that in both cases, $(P \to Q) \lor (Q \to R)$ holds. Then the main conclusion is reached by $\lor$ Elim and $\lor$ Intro.